ELEC0047 - Power system dynamics, control and stability

Dynamics of the synchronous machine

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Objective

- define accurately a number of time constants and inductances characterizing the machine electromagnetic transients
  - the latter appeared in the expression of the short-circuit current of the synchronous machine: see course ELEC0029
- use these expressions to derive from measurements the inductances and resistances of the Park model

Assumption

As we focus on electromagnetic transients, the rotor speed $\dot{\theta}$ is assumed constant, since it varies much more slowly.
Laplace transform of Park equations

\[
\begin{bmatrix}
V_d(s) + \dot{\theta}_r \psi_q(s) \\
- V_f(s) \\
0
\end{bmatrix}
= - \begin{bmatrix}
R_a + sL_{dd} & sL_{df} & sL_{dd1} \\
\phantom{R_a} & R_f + sL_{ff} & sL_{fd1} \\
\phantom{R_a} & \phantom{R_f} & R_{d1} + sL_{d1d1}
\end{bmatrix}
\begin{bmatrix}
I_d(s) \\
I_f(s) \\
I_{d1}(s)
\end{bmatrix}
+ \begin{bmatrix}
i_d(0) \\
i_f(0) \\
i_{d1}(0)
\end{bmatrix}
+ \begin{bmatrix}
L_d
\end{bmatrix}
\begin{bmatrix}
R_d + sL_d
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_q(s) - \dot{\theta}_r \psi_d(s) \\
0 \\
0
\end{bmatrix}
= - \begin{bmatrix}
R_a + sL_{qq} & sL_{qq1} & sL_{qq2} \\
\phantom{R_a} & R_{q1} + sL_{q1q1} & sL_{q1q2} \\
\phantom{R_a} & \phantom{R_{q1}} & R_{q2} + sL_{q2q2}
\end{bmatrix}
\begin{bmatrix}
I_q(s) \\
I_{q1}(s) \\
I_{q2}(s)
\end{bmatrix}
+ \begin{bmatrix}
i_q(0) \\
i_{q1}(0) \\
i_{q2}(0)
\end{bmatrix}
+ \begin{bmatrix}
L_q
\end{bmatrix}
\begin{bmatrix}
R_q + sL_q
\end{bmatrix}
\]
Time constants and inductances

Eliminating \( I_f, I_{d1}, I_{q1} \) and \( I_{q2} \) yields:

\[
V_d(s) + \dot{\theta}_r \psi_q(s) = -Z_d(s)I_d(s) + sG(s)V_f(s) \\
V_q(s) - \dot{\theta}_r \psi_d(s) = -Z_q(s)I_q(s)
\]

where:

\[
Z_d(s) = R_a + sL_{dd} - \left[ \begin{array}{cc} sL_{df} & sL_{dd1} \end{array} \right] \left[ \begin{array}{cc} R_f + sL_{ff} & sL_{fd1} \\ sL_{fd1} & R_d + sL_{d1d1} \end{array} \right]^{-1} \left[ \begin{array}{c} sL_{df} \\ sL_{dd1} \end{array} \right] = R_a + s\ell_d(s) \quad \ell_d(s) : d-axis operational inductance
\]

\[
Z_q(s) = R_a + sL_{qq} - \left[ \begin{array}{cc} sL_{qq1} & sL_{qq2} \end{array} \right] \left[ \begin{array}{cc} R_{q1} + sL_{q1q1} & sL_{q1q2} \\ sL_{q1q2} & R_{q2} + sL_{q2q2} \end{array} \right]^{-1} \left[ \begin{array}{c} sL_{qq1} \\ sL_{qq2} \end{array} \right] = R_a + s\ell_q(s) \quad \ell_q(s) : q-axis operational inductance
\]
Considering the nature of RL circuits, \( \ell_d(s) \) and \( \ell_q(s) \) can be factorized into:

\[
\ell_d(s) = L_{dd} \frac{(1 + sT'_d)(1 + sT''_d)}{(1 + sT'_{d0})(1 + sT''_{d0})}
\]

with \( 0 < T''_d < T'_{d0} < T'_d < T'_d \)

\[
\ell_q(s) = L_{qq} \frac{(1 + sT'_q)(1 + sT''_q)}{(1 + sT'_{q0})(1 + sT''_{q0})}
\]

with \( 0 < T''_q < T'_{q0} < T'_q < T'_q \)

Limit values:

\[
\lim_{s \to 0} \ell_d(s) = L_{dd} \quad \text{\textit{d-axis synchronous inductance}}
\]

\[
\lim_{s \to \infty} \ell_d(s) = L''_d = L_{dd} \frac{T'_d}{T'_{d0}} \frac{T''_d}{T''_{d0}} \quad \text{\textit{d-axis subtransient inductance}}
\]

\[
\lim_{s \to 0} \ell_q(s) = L_{qq} \quad \text{\textit{q-axis synchronous inductance}}
\]

\[
\lim_{s \to \infty} \ell_q(s) = L''_q = L_{qq} \frac{T'_q}{T'_{q0}} \frac{T''_q}{T''_{q0}} \quad \text{\textit{q-axis subtransient inductance}}
\]
Direct derivation of $L''_d$:

\[ R_d + sL_d \longrightarrow R_a + s\ell_d(s) \]

\[ s \to \infty \quad \downarrow \quad s \to \infty \]

\[ sL_d \longrightarrow sL''_d \]

elimin. of $f$ and $d_1$

\[
L''_d = L_{dd} - \left[ \begin{array}{cc} L_{df} & L_{dd_1} \\ \end{array} \right] \left[ \begin{array}{cc} L_{ff} & L_{fd_1} \\ L_{fd_1} & L_{dd_1} \end{array} \right]^{-1} \left[ \begin{array}{c} L_{df} \\ L_{dd_1} \end{array} \right] 
\]

\[ = L_{dd} - \frac{L_{df}^2 L_{dd_1} + L_{fd}^2 - 2L_{df} L_{fd_1} L_{dd_1}}{L_{ff} L_{dd_1} - L_{fd_1}^2} \]

and similarly for the q axis.
**Transient inductances**

If damper winding effects are neglected, the operational inductances simplify into:

\[ \ell_d(s) = L_{dd} \frac{1 + sT_d'}{1 + sT_{d0}'} \quad \ell_q(s) = L_{qq} \frac{1 + sT_q'}{1 + sT_{q0}'} \]

and the limit values become:

\[ \lim_{s \to \infty} \ell_d(s) = L_d' = L_{dd} \frac{T_d'}{T_{d0}'} \quad \text{d-axis transient inductance} \]

\[ \lim_{s \to \infty} \ell_q(s) = L_q' = L_{qq} \frac{T_q'}{T_{q0}'} \quad \text{q-axis transient inductance} \]

Using the same derivation as for \( L_d'' \), one easily gets:

\[ L_d' = L_{dd} - \frac{L_{df}^2}{L_{ff}} \quad L_q' = L_{qq} - \frac{L_{qq_1}^2}{L_{qq_1}q_1} \]
## Typical values

<table>
<thead>
<tr>
<th></th>
<th>machine with round rotor (pu)</th>
<th>machine with salient poles (pu)</th>
<th>machine with round rotor (s)</th>
<th>machine with salient poles (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_d$</td>
<td>1.5-2.5</td>
<td>0.9-1.5</td>
<td>$T'_d$</td>
<td>8.0-12.0</td>
</tr>
<tr>
<td>$L_q$</td>
<td>1.5-2.5</td>
<td>0.5-1.1</td>
<td>$T'_d$</td>
<td>0.95-1.30</td>
</tr>
<tr>
<td>$L'_d$</td>
<td>0.2-0.4</td>
<td>0.3-0.5</td>
<td>$T'_d$</td>
<td>0.02-0.05</td>
</tr>
<tr>
<td>$L'_q$</td>
<td>0.2-0.4</td>
<td>0.25-0.35</td>
<td>$T'_q$</td>
<td>2.0</td>
</tr>
<tr>
<td>$L''_d$</td>
<td>0.15-0.30</td>
<td>0.25-0.35</td>
<td>$T''_d$</td>
<td>0.8</td>
</tr>
<tr>
<td>$L''_q$</td>
<td>0.15-0.30</td>
<td>0.25-0.35</td>
<td>$T''_d$</td>
<td>0.20-0.50</td>
</tr>
</tbody>
</table>

Inductances in per unit on the machine nominal voltage and apparent power.
Comments

- in the direct axis: pronounced “time decoupling”:

\[
T_{d0}^{'} \gg T_{d0}^{''} \quad T_{d}^{'} \gg T_{d}^{''}
\]

- subtransient time constants \(T_{d}^{''}\) and \(T_{d0}^{''}\): short, originate from damper winding
- transient time constants \(T_{d}^{'}\) and \(T_{d0}^{'}\): long, originate from field winding

- in the quadrature axis: less pronounced time decoupling
  - because the windings are of quite different nature!

- salient-pole machines: single winding in q axis ⇒ the parameters \(L_{q}^{'}\), \(T_{q}^{'}\) and \(T_{q0}^{'}\) do not exist.
Rotor motion

$\theta_m$ angular position of rotor, i.e. angle between one axis attached to the rotor and one attached to the stator. Linked to “electrical” angle $\theta_r$ through:

$$\theta_r = p \theta_m \quad \text{p number of pairs of poles}$$

$\omega_m$ mechanical angular speed:

$$\omega_m = \frac{d}{dt} \theta_m$$

$\omega$ electrical angular speed:

$$\omega = \frac{d}{dt} \theta_r = p \omega_m$$

Basic equation of rotating masses (friction torque neglected):

$$I \frac{d}{dt} \omega_m = T_m - T_e$$

$I$ moment of inertia of all rotating masses

$T_m$ mechanical torque provided by prime mover (turbine, diesel motor, etc.)

$T_e$ electromagnetic torque developed by synchronous machine
Motion equation expressed in terms of $\omega$:

$$\frac{l}{p} \frac{d}{dt} \omega = T_m - T_e$$

Dividing by the base torque $T_B = S_B/\omega_{mB}$:

$$\frac{l\omega_{mB}}{pS_B} \frac{d}{dt} \omega = T_{mpu} - T_{epu}$$

Defining the speed in per unit:

$$\omega_{pu} = \frac{\omega}{\omega_N} = \frac{1}{\omega_N} \frac{d}{dt} \theta_r$$

and taking $\omega_{mB} = \omega_B/p = \omega_N/p$, the motion equation becomes:

$$\frac{l\omega^2_{mB}}{S_B} \frac{d}{dt} \omega_{pu} = T_{mpu} - T_{epu}$$

Defining the inertia constant:

$$H = \frac{1}{2} \frac{l\omega^2_{mB}}{S_B}$$

the motion equation is rewritten as:

$$2H \frac{d}{dt} \omega_{pu} = T_{mpu} - T_{epu}$$
**Inertia constant $H$**

- called *specific energy*
- ratio \( \frac{\text{kinetic energy of rotating masses at nominal speed}}{\text{apparent nominal power of machine}} \)
- has dimension of a time
- with values in rather narrow interval, whatever the machine power.

<table>
<thead>
<tr>
<th>$H$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>thermal plant</td>
<td>hydro plant</td>
</tr>
<tr>
<td>$p = 1 : 2$</td>
<td>$1.5 - 3$ s</td>
</tr>
<tr>
<td>$p = 2 : 3$</td>
<td>$7$ s</td>
</tr>
</tbody>
</table>
Relationship between $H$ and launching time $t_l$

$t_l$ : time to reach the nominal angular speed $\omega_{mB}$ when applying to the rotor, initially at rest, the nominal mechanical torque:

$$T_N = \frac{P_N}{\omega_{mB}} = \frac{S_B \cos \phi_N}{\omega_{mB}}$$

$P_N$: turbine nominal power (in MW) \hspace{1cm} \cos \phi_N$: nominal power factor

Nominal mechanical torque in per unit:

$$T_{Npu} = \frac{T_N}{T_B} = \cos \phi_N$$

Uniformly accelerated motion:

$$\omega_{mpu} = \omega_{mpu}(0) + \frac{\cos \phi_N}{2H} t = \frac{\cos \phi_N}{2H} t$$

At $t = t_l$, $\omega_{mpu} = 1$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $t_l = \frac{2H}{\cos \phi_N}$

Remark. Some define $t_l$ with reference to $T_B$, not $T_N$. In this case, $t_l = 2H$. 

Compensated motion equation

In some simplified models, the damper windings are neglected. To compensate for the neglected damping torque, a correction term can be added:

\[ 2H \frac{d}{dt} \omega_{pu} + D(\omega_{pu} - \omega_{sys}) = T_{mpu} - T_{epu} \quad D \geq 0 \]

where \( \omega_{sys} \) is the system angular frequency (which will be defined in “Power system dynamic modelling under the phasor approximation”).

Expression of electromagnetic torque

\[ T_e = p(\psi_d i_q - \psi_q i_d) \]

Using the base defined in slide \# 16:

\[ T_{epu} = \frac{T_e}{T_B} = \frac{\omega_{mb} S_B}{\sqrt{3}V_B \sqrt{3}I_B} p(\psi_d i_q - \psi_q i_d) = \frac{\omega_B}{\sqrt{3}V_B \sqrt{3}I_B} (\psi_d i_q - \psi_q i_d) \]

\[ = \frac{\psi_d}{\sqrt{3}V_B} \frac{i_q}{\sqrt{3}I_B} - \frac{\psi_q}{\sqrt{3}V_B} \frac{i_d}{\sqrt{3}I_B} = \psi_{dpu} i_{qpu} - \psi_{qpu} i_{dpu} \]

In per unit, the factor \( p \) disappears.
Recall on per unit systems

Consider two magnetically coupled coils with:

\[ \psi_1 = L_{11} i_1 + L_{12} i_2 \quad \psi_2 = L_{21} i_1 + L_{22} i_2 \]

For simplicity, we take the same time base in both circuits: \( t_{1B} = t_{2B} \)

In per unit:

\[
\psi_{1pu} = \frac{\psi_1}{\psi_{1B}} = L_{11} \frac{i_1}{L_{1B} i_{1B}} + L_{12} \frac{i_2}{L_{1B} i_{1B}} = L_{11pu} i_{1pu} + L_{12} \frac{i_{2pu}}{L_{1B} i_{1B}} \\
\psi_{2pu} = \frac{\psi_2}{\psi_{2B}} = L_{21} \frac{i_{1pu}}{L_{2B} i_{2B}} + L_{22} i_{2pu}
\]

In Henry, one has \( L_{12} = L_{21} \). We request to have the same in per unit:

\[
L_{12pu} = L_{21pu} \iff \frac{i_{2B}}{L_{1B} i_{1B}} = \frac{i_{1B}}{L_{2B} i_{2B}} \iff S_{1B} t_{1B} = S_{2B} t_{2B} \iff S_{1B} = S_{2B}
\]

A per unit system with \( t_{1B} = t_{2B} \) and \( S_{1B} = S_{2B} \) is called \textit{reciprocal}.
<table>
<thead>
<tr>
<th></th>
<th>in the single phase circuit equivalent to stator windings</th>
<th>in each of the $d, q$ windings</th>
<th>in each rotor winding, for instance $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td></td>
<td>$t_B = \frac{1}{\omega_N} = \frac{1}{2\pi f_N}$</td>
<td></td>
</tr>
<tr>
<td>power</td>
<td></td>
<td>$S_B = \text{nominal apparent 3-phase}$</td>
<td></td>
</tr>
<tr>
<td>voltage</td>
<td>$V_B$: nominal (rms) phase-neutral</td>
<td>$\sqrt{3}V_B$</td>
<td>$V_{fB}$: to be chosen</td>
</tr>
<tr>
<td>current</td>
<td>$I_B = \frac{S_B}{3V_B}$</td>
<td>$\sqrt{3}I_B$</td>
<td>$\frac{S_B}{V_{fB}}$</td>
</tr>
<tr>
<td>impedance</td>
<td>$Z_B = \frac{3V_B^2}{S_B}$</td>
<td>$\frac{3V_B^2}{S_B}$</td>
<td>$\frac{V_{fB}^2}{S_B}$</td>
</tr>
<tr>
<td>flux</td>
<td>$V_B t_B$</td>
<td>$\sqrt{3}V_B t_B$</td>
<td>$V_{fB} t_B$</td>
</tr>
</tbody>
</table>
The equal-mutual-flux-linkage per unit system

For two magnetically coupled coils, it is shown that (see theory of transformer):

\[
\begin{align*}
L_{11} - L_{\ell 1} &= \frac{n_1^2}{R} \\
L_{12} &= \frac{n_1 n_2}{R} \\
L_{22} - L_{\ell 2} &= \frac{n_2^2}{R}
\end{align*}
\]

- \(L_{11}\) self-inductance of coil 1
- \(L_{22}\) self-inductance of coil 1
- \(L_{\ell 1}\) leakage inductance of coil 1
- \(L_{\ell 2}\) leakage inductance of coil 2
- \(n_1\) number of turns of coil 1
- \(n_2\) number of turns of coil 2
- \(R\) reluctance of the magnetic circuit followed by the magnetic field lines which cross both windings; the field is created by \(i_1\) and \(i_2\).
Assume we choose $V_{1B}$ and $V_{2B}$ such that:

$$\frac{V_{1B}}{V_{2B}} = \frac{n_1}{n_2}$$

In order to have the same base power in both circuits:

$$V_{1B}I_{1B} = V_{2B}I_{2B} \quad \Rightarrow \quad \frac{I_{1B}}{I_{2B}} = \frac{n_2}{n_1}$$

We have:

$$(L_{11} - L_{\ell 1})I_{1B} = \frac{n_1^2}{R} I_{1B} = \frac{n_2}{n_1} I_{2B} = \frac{n_1n_2}{R} I_{2B} = L_{12}I_{2B} \quad (1)$$

The flux created by $I_{2B}$ in coil 1 is equal to the flux created by $I_{1B}$ in the same coil 1, after removing the part that corresponds to leakages.

Similarly in coil 2:

$$(L_{22} - L_{\ell 2})I_{2B} = \frac{n_2^2}{R} I_{2B} = \frac{n_1}{n_2} I_{1B} = \frac{n_1n_2}{R} I_{1B} = L_{12}I_{1B} \quad (2)$$

This per unit system is said to yield equal mutual flux linkages (EMFL)
Alternative definition of base currents

From respectively (1) and (2):

\[
\frac{I_{1B}}{I_{2B}} = \frac{L_{12}}{L_{11} - L_{\ell 1}} \quad \frac{I_{1B}}{I_{2B}} = \frac{L_{22} - L_{\ell 2}}{L_{12}}
\]

A property of this pu system

\[
L_{12pu} = \frac{L_{12}I_{2B}}{L_{1B}I_{1B}} = \frac{(L_{11} - L_{\ell 1})}{L_{1B}} = L_{11pu} - L_{\ell 1pu} \quad (3)
\]

\[
L_{21pu} = \frac{L_{21}I_{1B}}{L_{2B}I_{2B}} = \frac{(L_{22} - L_{\ell 2})}{L_{2B}} = L_{22pu} - L_{\ell 2pu}
\]

In this pu system, self-inductance = mutual inductance + leakage reactance. Does not hold true for inductances in Henry!

The inductance matrix of the two coils takes on the form:

\[
L = \begin{bmatrix}
L_{11} & L_{12} \\
L_{12} & L_{22}
\end{bmatrix} = \begin{bmatrix}
L_{\ell 1} + M & M \\
M & L_{\ell 2} + M
\end{bmatrix}
\]
Application to synchronous machine

- we have to choose a base voltage (or current) in each rotor winding. Let’s first consider the field winding \( f \) (1 \( \equiv f \), 2 \( \equiv d \))
- we would like to use the EMFL per unit system
- we do not know the “number of turns” of the equivalent circuits \( f \), \( d \), etc.
- instead, we can use one of the alternative definitions of base currents:

\[
\frac{I_{fB}}{\sqrt{3}I_B} = \frac{L_{dd} - L_\ell}{L_{df}} \quad \Rightarrow \quad I_{fB} = \sqrt{3}I_B \frac{L_{dd} - L_\ell}{L_{df}}
\]  

(4)

- \( L_{dd}, L_\ell \) can be measured
- \( L_{df} \) can be obtained by measuring the no-load voltage \( E_q \) produced by a known field current \( i_f \):

\[
E_q = \frac{\omega_N L_{df}}{\sqrt{3}} i_f \quad \Rightarrow \quad L_{df} = \frac{\sqrt{3}E_q}{\omega_N i_f}
\]

(5)

- the base voltage is obtained from \( V_{fB} = \frac{S_B}{I_{fB}} \)
What about the other rotor windings?

- one cannot access the $d_1$, $q_1$ and $q_2$ windings to measure $L_{dd1}$, $L_{qq1}$ et $L_{qq2}$ using formulae similar to (5)
- it can be assumed that there exist base currents $I_{d1B}$, $I_{q1B}$ et $I_{q2B}$ leading to the EMFL per unit system, but their values are not known
- hence, we cannot compute voltages in Volt or currents in Ampere in those windings (only in pu)
- not a big issue in so far as we do not have to connect anything to those windings (unlike the excitation system to the field winding)...
Numerical example

A machine has the following characteristics:

- nominal frequency: 50 Hz
- nominal apparent power: 1330 MVA
- stator nominal voltage: 24 kV
- $X_d = 0.9 \, \Omega$  \quad $X_\ell = 0.1083 \, \Omega$
- field current giving the nominal stator voltage at no-load (and nominal frequency): 2954 A

1. Base power, voltage, impedance, inductance and current at the stator

\[ S_B = 1330 \, \text{MVA} \]
\[ U_B = 24000 \, \text{V} \]
\[ V_B = \frac{24000}{\sqrt{3}} = 13856 \, \text{V} \]
\[ Z_B = \frac{3 \, V_B^2}{S_B} = 0.4311 \, \Omega \quad \Rightarrow \quad L_B = \frac{Z_B}{\omega_B} = \frac{Z_B}{\omega_N} = \frac{Z_B}{2 \pi 50} = 1.378 \times 10^{-3} \, \text{H} \]
\[ I_B = \frac{S_B}{3 \, V_B} = \frac{1330 \times 10^6}{3 \times 13856} = 31998 \, \text{A} \]
2. Base power, current, voltage, impedance and induct. in field winding

\[ S_{fB} = S_B = 1330 \text{ MVA} \]

\[ L_{df} = \frac{\sqrt{3} E_q}{\omega_N i_f} = \frac{\sqrt{3} \left( \frac{24}{\sqrt{3}} \right) 10^3}{2 \pi \frac{50}{2954}} = 2.586 \times 10^{-2} \text{ H} \]

\[ L_{dd} = \frac{X_d}{\omega_N} = \frac{0.9}{2 \pi \frac{50}{2}} = 2.865 \times 10^{-3} \text{ H} \]

\[ L_\ell = \frac{X_\ell}{\omega_N} = \frac{0.1083}{2 \pi \frac{50}{2}} = 3.447 \times 10^{-4} \text{ H} \]

Using Eq. (4):
\[ I_{fB} = \sqrt{3} I_B \frac{L_{dd} - L_\ell}{L_{df}} = 5401 \text{ A} \]

\[ V_{fB} = \frac{S_{fB}}{I_{fB}} = \frac{1330 \times 10^6}{5401} = 246251 \text{ V} \]

A huge value! This is to be expected since we use the machine nominal power \( S_B \) in the field winding, which is not designed to carry such a high power!

\[ Z_{fB} = \frac{V_{fB}}{I_{fB}} = \frac{246251}{5401} = 45.594 \Omega \quad \Rightarrow \quad L_{fB} = \frac{Z_{fB}}{\omega_B} = \frac{45.594}{2 \pi \frac{50}{2}} = 0.14513 \text{ H} \]
3. Convert $L_{dd}$ and $L_{df}$ in per unit

\[ L_{dd} = X_d = \frac{0.9}{0.4331} = 2.078 \text{ pu} \]
\[ L_{\ell} = \frac{0.1083}{0.4331} = 0.25 \text{ pu} \]

In per unit, in the EMFL per unit system, Eq. (3) can be used. Hence:

\[ L_{df} = L_{dd} - L_{\ell} = 2.078 - 0.25 = 1.828 \text{ pu} \] \hspace{1cm} (6)

Remarks

- Eq. (6) does not hold true in Henry:

\[ L_{dd} - L_{\ell} = 2.865 \times 10^{-3} - 3.447 \times 10^{-4} = 2.5203 \times 10^{-3} \text{ H} \neq L_{df} = 2.586 \times 10^{-2} \text{ H} \]

- in the EMFL per unit system, $L_{dd}$ and $L_{df}$ have comparable values
In the EMFL per unit system, the Park inductance matrices take on the simplified form:

\[
L_d = \begin{bmatrix}
L_\ell + M_d & M_d & M_d \\
M_d & L_\ell f + M_d & M_d \\
M_d & M_d & L_\ell d_1 + M_d \\
\end{bmatrix}
\]

\[
L_q = \begin{bmatrix}
L_\ell + M_q & M_q & M_q \\
M_q & L_\ell q_1 + M_q & M_q \\
M_q & M_q & L_\ell q_2 + M_q \\
\end{bmatrix}
\]

For symmetry reasons, same leakage inductance \( L_\ell \) in \( d \) and \( q \) windings.
Saturation of magnetic material modifies:
- the machine inductances
- the initial operating point (in particular the rotor position)
- the field current required to obtain a given stator voltage.

**Notation**
- parameters with the upperscript $u$ refer to *unsaturated* values
- parameters without this upperscript refer to *saturated* values.
Open-circuit magnetic characteristic

Machine operating at no load, rotating at nominal angular speed $\omega_N$. Terminal voltage $E_q$ measured for various values of the field current $i_f$.

saturation factor: $k = \frac{OA}{OB} = \frac{O'A'}{O'A} < 1$

a standard model: $k = \frac{1}{1 + m(E_q)^n}$ $m, n > 0$

characteristic in $d$ axis (field due to $i_f$ only)

In per unit: $E_{qpu} = \frac{\omega_N L_{df} i_f}{\sqrt{3} V_B} = \frac{\omega_N L_{df} I_{fB}}{\sqrt{3} V_B} i_{fpu} = \frac{\omega_N L_{df}}{\sqrt{3} V_B} \sqrt{3} I_B \frac{L_{dd} - L_\ell}{L_{df}} i_{fpu}$

$= \frac{\omega_N}{Z_B} (L_{dd} - L_\ell) i_{fpu} = M_{dpu} i_{fpu}$

Dropping the $pu$ notation and introducing $k$:

$E_q = M_d i_f = k M^u_d i_f$
Leakage and air gap flux

The flux linkage in the $d$ winding is decomposed into:

$$\psi_d = L_\ell i_d + \psi_{ad}$$

$L_\ell i_d$: leakage flux, not subject to saturation (path mainly in the air)  
$\psi_{ad}$: direct-axis component of the air gap flux, subject to saturation.

Expression of $\psi_{ad}$:

$$\psi_{ad} = \psi_d - L_\ell i_d = M_d(i_d + i_f + i_{d1})$$

Expression of $\psi_{aq}$:

$$\psi_{aq} = \psi_q - L_\ell i_q = M_q(i_q + i_{q1} + i_{q2})$$

Considering that the $d$ and $q$ axes are orthogonal, the air gap flux is given by:

$$\psi_{ag} = \sqrt{\psi_{ad}^2 + \psi_{aq}^2}$$
Saturation characteristic in loaded conditions

- Saturation is different in the \( d \) and \( q \) axes, especially for a salient pole machine (air gap larger in \( q \) axis!). Hence, different saturation factors (say, \( k_d \) and \( k_q \)) should be considered.
- In practice, however, it is quite common to have only the direct-axis saturation characteristic.
- In this case, the characteristic is used along any axis (not just \( d \)) as follows.
- In no-load conditions, we have

\[
\psi_{ad} = M_d i_f \quad \text{and} \quad \psi_{aq} = 0 \quad \Rightarrow \quad \psi_{ag} = M_d i_f
\]

\[
M_d = k M_d^u = \frac{M_d^u}{1 + m(E_q)^n} = \frac{M_d^u}{1 + m(M_d i_f)^n} = \frac{M_d^u}{1 + m(\psi_{ag})^n}
\]

- It is assumed that this relation still holds true with the combined air gap flux \( \psi_{ag} \) given by (7).
Dynamics of the synchronous machine

Summary: complete model (variables in per unit)

\[
\begin{align*}
\psi_d &= L\ell i_d + \psi_{ad} \\
\psi_f &= L\ell f i_f + \psi_{ad} \\
\psi_{d1} &= L\ell_{d1} i_{d1} + \psi_{ad} \\
\psi_{ad} &= M_d (i_d + i_f + i_{d1}) \\
M_d &= \frac{M_d^u}{1 + m \left( \sqrt{\psi_{ad}^2 + \psi_{aq}^2} \right)^n} \\
v_d &= -R_a i_d - \omega \psi_q - \frac{d\psi_d}{dt} \\
\psi_q &= L\ell i_q + \psi_{aq} \\
\psi_{q1} &= L\ell_{q1} i_{q1} + \psi_{aq} \\
\psi_{q2} &= L\ell_{q2} i_{q2} + \psi_{aq} \\
\psi_{aq} &= M_q (i_q + i_{q1} + i_{q2}) \\
M_q &= \frac{M_q^u}{1 + m \left( \sqrt{\psi_{ad}^2 + \psi_{aq}^2} \right)^n} \\
v_q &= -R_a i_q + \omega \psi_d - \frac{d\psi_q}{dt} \\
\frac{d}{dt} \psi_f &= v_f - R_f i_f \\
\frac{d}{dt} \psi_{d1} &= -R_{d1} i_{d1} \\
\frac{d}{dt} \psi_{q1} &= -R_{q1} i_{q1} \\
\frac{d}{dt} \psi_{q2} &= -R_{q2} i_{q2} \\
2H \frac{d}{dt} \omega &= T_m - (\psi_d i_q - \psi_q i_d) \\
\frac{1}{\omega_N} \frac{d}{dt} \theta_r &= \omega
\end{align*}
\]
Model simplifications

Constant rotor speed approximation \( \dot{\theta}_r \simeq \omega_N \) \((\omega = 1 \text{ pu})\)

Examples showing that \( \dot{\theta}_r \) does not depart much from the nominal value \( \omega_N \):

1. Oscillation of \( \theta_r \) with a magnitude of 90° and period of 1 second superposed to the uniform motion at synchronous speed:

\[
\theta_r = \theta^o_r + 2\pi f_N t + \frac{\pi}{2} \sin 2\pi t \quad \Rightarrow \quad \dot{\theta}_r = 100\pi + \pi^2 \cos 2\pi t \simeq 314 + 10 \cos 2\pi t
\]

At its maximum, it deviates from nominal by \(10/314 = 3\%\) only.

2. In a large interconnected system, after primary frequency control, frequency settles at \( f \neq f_N \). \(|f - f_N| = 0.1 \text{ Hz}\) is already a large deviation. In this case, machine speeds deviate from nominal by \(0.1/50 = 0.2\%\) only.

3. A small isolated system may experience larger frequency deviations. But even for \(|f - f_N| = 1 \text{ Hz}\), the machine speeds deviate from nominal by \(1/50 = 2\%\) only.
The phasor (or quasi-sinusoidal) approximation

- Underlies a large class of power system dynamic simulators
- Considered in detail in the following lectures
- For the synchronous machine, it consists of neglecting the “transformer voltages” $\frac{d\psi_d}{dt}$ and $\frac{d\psi_q}{dt}$ in the stator Park equations
- This leads to neglecting some fast varying components of the network response, and allows the voltage and currents to be treated as a sinusoidal with time-varying amplitudes and phase angles (hence the name)
- At the same time, three-phase balance is also assumed.

Thus, the stator Park equations become (in per unit, with $\omega = 1$):

$$v_d = -R_a i_d - \psi_q$$
$$v_q = -R_a i_q + \psi_d$$

And $\psi_d$ and $\psi_q$ are now algebraic, instead of differential, variables.

Hence, they may undergo a discontinuity after of a network disturbance.
The “classical” model of the synchronous machine

Very simplified model used:
- in some analytical developments
- in qualitative reasoning dealing with transient (angle) stability
- for fast assessment of transient (angle) stability.

“Classical” refers to a model used when there was little computational power.

Approximation # 0. We consider the phasor approximation.

Approximation # 1. The damper windings $d_1$ et $q_2$ are ignored.
- The damping of rotor oscillations is going to be underestimated.

Approximation # 2. The stator resistance $R_a$ is neglected.
- This is very acceptable.

The stator Park equations become:

$$
\begin{align*}
\nu_d &= -\psi_q = -L_{qq}i_q - L_{qq_1}i_{q_1} \\
\nu_q &= \psi_d = L_{dd}i_d + L_{df}i_f
\end{align*}
$$
Expressing $i_f$ (resp. $i_{q1}$) as function of $\psi_f$ and $i_d$ (resp. $\psi_{q1}$ and $i_q$):

$$
\begin{align*}
\psi_f &= L_{ff} i_f + L_{df} i_d \implies i_f = \frac{\psi_f - L_{df} i_d}{L_{ff}} \\
\psi_{q1} &= L_{q1q1} i_{q1} + L_{qq} i_{q1} \implies i_{q1} = \frac{\psi_{q1} - L_{qq} i_q}{L_{q1q1}}
\end{align*}
$$

and introducing into the stator Park equations:

$$
\begin{align*}
  v_d &= -(L_{qq} - \frac{L_{qq}^2}{L_{q1q1}}) i_q - \frac{L_{qq}}{L_{q1q1}} \psi_{q1} = -X'_{q} i_q + e'_d \\
  v_q &= (L_{dd} - \frac{L_{df}^2}{L_{ff}}) i_d + \frac{L_{df}}{L_{ff}} \psi_f = X'_{d} i_d + e'_q
\end{align*}
$$

$e'_d$ and $e'_q$:

- are called the *e.m.f. behind transient reactances*
- are proportional to magnetic fluxes; hence, they cannot vary much after a disturbance, unlike the rotor currents $i_f$ and $i_{q1}$.
Approximation # 3. The e.m.f. $e'_d$ and $e'_q$ are assumed constant.

- This is valid over no more than - say - one second after a disturbance;
- over this interval, a single rotor oscillation can take place; hence, damping cannot show its effect (i.e. Approximation # 1 is not a concern).

Equations (8, 9) are similar to the Park equations in steady state, except for the presence of an e.m.f. in the $d$ axis, and the replacement of the synchronous by the transient reactances.

Approximation # 4. The transient reactance is the same in both axes: $X'_d = X'_q$.

- Questionable, but experiences shows that $X'_q$ has less impact . . .

If $X'_d = X'_q$, Eqs. (8, 9) can be combined in a single phasor equation, with the corresponding equivalent circuit:

\[
\vec{V} + jX'_d \vec{I} = \vec{E'} = E' \angle \delta
\]

\[
\vec{V} = V \angle \theta
\]
Rotor motion. This is the only dynamics left!

e'_d and e'_q are constant. Hence, $\bar{E}'$ is fixed with respect to $d$ and $q$ axes, and $\delta$ differs from $\theta_r$ by a constant.

Therefore, \[ \frac{1}{\omega_N} \frac{d}{dt} \theta_r = \omega \quad \text{can be rewritten as:} \quad \frac{1}{\omega_N} \frac{d}{dt} \delta = \omega \]

The rotor motion equation:

\[ 2H \frac{d}{dt} \omega = T_m - T_e \]

is transformed to involve powers instead of torques. Multiplying by \( \omega \):

\[ 2H \omega \frac{d}{dt} \omega = \omega T_m - \omega T_e \]

\( \omega T_m = \text{mechanical power } P_m \text{ of the turbine} \)

\( \omega T_e = p_{r\rightarrow s} = p_T(t) + p_{Js} + \frac{dW_{ms}}{dt} \approx P \) (active power produced)

since we assume three-phase balanced AC operation, and \( R_a \) is neglected
Approximation # 5. We assume $\omega \approx 1$ and replace $2H\omega$ by $2H$
- very acceptable, already justified.

Thus we have:

$$2H \frac{d}{dt}\omega = P_m - P$$

where $P$ can be derived from the equivalent circuit:

$$P = \frac{E' V}{X'} \sin(\delta - \theta)$$